A New Approach to Estimate Elasticity of Marginal Utility of Income

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Road Map

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Introduction of the Issue

- The income elasticity of marginal utility of income (σ) and price elasticities (η_{ij}) are key input parameters for various micro and macro policy analyses, Computable General Equilibrium (CGE) modelling and public economics decision making.
- Generally, the income elasticity of marginal utility of income and price elasticities are calculated by estimating demand systems using the complete quantity, price and income data.
- However, most of the Household Expenditure Surveys (HES) collect and publish only expenditure data and do not have separate quantity or price data to estimate the demand systems and hence it is not possible to estimate σ and η_{ij}.
- This poses substantial challenges in using the HES data for consumer demand analysis.

Introduction of the Issue

- When price data are not available, it is common practice to specify a value for σ = -2, a common value for σ reported in the literature when price data are available, to calculate the price elasticities, η_{ij}.
- For example, Clements et al. (2022) used Australian HES data to estimate price elasticities when price data are not available, but specifying that σ = -2.

Consumer Maximization Problem

- Let q_{ih} and p_{ih} are the consumption and price of commodity i by household h and $u(q_{1h}, q_{2h}, \dots, q_{nh})$ is the household utility.

- The consumer budget is $M_h = \sum_{i=1}^{n} p_i\, q_{ih}$

- The Lagrangian function for the maximisation of the utility subject to the budget constraint is

$$u^*(q_{1h}, q_{2h}, \dots, q_{nh}, \lambda) = u(q_{1h}, q_{2h}, \dots, q_{nh}) + \lambda\left(M_h - \sum_{i=1}^{n} p_i\, q_{ih}\right)$$

Consumer Maximization Problem

- The first-order conditions for the above maximisation problem are the budget constraint and

$$\frac{\partial u}{\partial q_i} = \lambda p_i, \qquad \text{i=1,2,…, n.}$$

- Solving the first order conditions we get a system consisting of n demand equations of the form

$$q_i = q_i(M, p_1, p_2, \ldots, p_n), \qquad \text{i=1,2,…, n.}$$

The Linear Expenditure System (LES)

- The utility function of household *h* behind the LES is based on the additive utility function (Stone, 1954),

$$U(q_{1h}, q_{2h}, \dots, q_{nh}) = \left(\sum_{i=1}^{n} \beta_i \log(q_{ih} - \gamma_{ih}) \right),$$

where q_{ih} is the consumption of commodity *i* by household *h*, the coefficients β_i and γ_{ih} satisfy $\sum_{i=1}^{n} \beta_i = 1$, $0 < \beta_i < 1$; and $q_{ih} > \gamma_{ih}$, $i=1,2, \dots, n$.

The Linear Expenditure System (LES)

$$p_i q_{ih} = p_i \gamma_{ih} + \beta_i \left(\mathrm{M}_h - \sum_{j=1}^{n} p_j \gamma_{jh} \right), \qquad i=1,2,\ldots,n.$$

$$\lambda = \frac{1}{\left(\mathrm{M}_h - \sum_{j=1}^{n} p_j \gamma_{jh} \right)}.$$

Income elasticity of marginal utility of income

- The income flexibility ϕ_h (or the inverse of the income elasticity of marginal utility of income σ_h) based on LES is given by
  - $\sigma = \left[\frac{\partial(\log \lambda)}{\partial(\log M)}\right]$

$$\phi_h = \left[\frac{\partial(\log \lambda)}{\partial(\log M_h)}\right]^{-1} = \left(\sum_{j=1}^{n} s_{jh}\right) - 1.$$

- That is,

$$\left(\sum_{j=1}^{n} s_{jh}\right) = 1 + \phi_h = 1 + (1/\sigma_h).$$

where $s_{ih} = p_i \gamma_{ih} / M_h$,

Income and Price Elasticities from LES

- The income elasticity (η_i):

$$\eta_i = \frac{\partial(\log\ q_{ih})}{\partial(\log M_h)} = \frac{\beta_i}{w_{ih}}, \qquad\qquad i,j=1,2,\ldots, n;$$

- Marshallian (uncompensated) price elasticities (η_{ij}^*):

$$\eta_{ij}^* = \frac{\partial(\log q_{ih})}{\partial(\log p_j)} = \delta_{ij}\left(\frac{s_{ih}}{w_{ih}} - 1\right) - \frac{\beta_i}{w_{ih}} s_{jh}, \qquad\qquad i,j=1,2,\ldots, n;$$

Income and Price Elasticities from LES

- Slutsky (compensated) price elasticities (η_{ij}):

$$\eta_{ij} = \eta^*_{ij} + w_{jh}\,\eta_i = \delta_{ij}\left(\left(\frac{s_{ih}}{w_{ih}} - 1\right)\right) + \frac{\beta_i}{w_{ih}}(w_{jh} - s_{jh}), \quad i,j=1,2,\ldots,n,$$

where $w_{ih} = p_i q_{ih} / M_h$ is the budget share and $s_{ih} = p_i \gamma_{ih} / M_h$, is the subsistence share of i of household h and δ_{ij} is the Kronecker delta.

A New Method to Estimate σ_h

- *Step 1:*

Linear Expenditure System (LES)

$$p_i q_{ih} = p_i \gamma_{ih} + \beta_i \left(\mathrm{M}_h - \sum_{j=1}^{n} p_j \gamma_{jh} \right), \qquad i=1,2,\ \ldots,\ n.$$

$$p_i q_{ih} = \alpha_i + \beta_i M_h, \qquad i=1,2,\ \ldots,\ n, \qquad (1)$$

where

$$\alpha_i = p_i \gamma_{ih} - \beta_i \sum_{j=1}^{n} p_j \gamma_{jh}\,, \quad i=1,2,\ \ldots,\ n. \qquad (2)$$

The coefficients satisfy

$$\sum_{i=1}^{n} \alpha_i = 0 \qquad \text{and} \qquad \sum_{i=1}^{n} \beta_i = 1.$$

Estimate equation (1) and obtain estimates β_i, *i=1,2, …, n*.

A New Method to Estimate σ_h

- *Step 2:*
  Linear Expenditure System (LES)

$$p_i q_{ih} = p_i \gamma_{ih} + \beta_i \left(\mathrm{M}_h - \sum_{j=1}^{n} p_j \gamma_{jh}\right), \quad i=1,2,\ldots,n.$$

Dividing both sides of LES by M_h, we get

$$w_{ih} = s_{ih} + \phi_h(-\beta_i), \qquad i=1,2,\ldots,n. \qquad (3)$$

Using estimates for β_i from (1) and estimate equation (3) to obtain an estimate for ϕ_h. Then rearranging (2) and dividing it by M_h, we can calculate s_{ih} as

$$\alpha_i = p_i \gamma_{ih} - \beta_i \sum_{j=1}^{n} p_j \gamma_{jh}, \qquad i=1,2,\ldots,n.$$

(2)

$$s_{ih} = (\alpha_i / M_h) + \beta_i (1 + \phi_h)$$

A New Method to Estimate σ_h

- *Step 3:*

The income elasticity (η_i),

$$\eta_i = \frac{\partial(\log\ q_{ih})}{\partial(\log M_h)} = \frac{\beta_i}{w_{ih}}, \qquad i,j=1,2,\ldots,n;$$

Marshallian (uncompensated) price elasticities (η^*_{ij}):

$$\eta^*_{ij} = \frac{\partial(\log q_{ih})}{\partial(\log p_j)} = \delta_{ij}\left(\frac{s_{ih}}{w_{ih}} - 1\right) - \frac{\beta_i}{w_{ih}} s_{jh}, \qquad i,j=1,2,\ldots,n;$$

Slutsky (compensated) price elasticities (η_{ij}):

$$\eta_{ij} = \eta^*_{ij} + w_{jh}\,\eta_i = \delta_{ij}\left(\left(\frac{s_{ih}}{w_{ih}} - 1\right)\right) + \frac{\beta_i}{w_{ih}}(w_{jh} - s_{jh}),$$

$i,j=1,2,\ldots,n$.

A New Method to Estimate σ_h

Table 1:

Estimated mean income flexibility (ϕ_h) and income elasticity of marginal utility of income (σ_h) by income group, Sri Lanka

Income group (1)	Number of estimates (2)	ϕ_h (3)	Standard error (4)	σ_h (5)	Standard error (6)
Quintile 1 (poorest)	2,613	-0.192	0.004	-8.945	0.104
Quintile 2	2,613	-0.277	0.005	-6.747	0.098
Quintile 3	2,613	-0.365	0.006	-5.269	0.089
Quintile 4	2,613	-0.470	0.006	-4.009	0.075
Quintile 5 (richest)	2,612	-0.674	0.008	-2.748	0.060
Full sample	13,064	-0.396	0.003	-5.544	0.043

A New Method to Estimate σ_h

Table 2:

Income and price elasticity estimates for Sri Lanka

Commodity (1)	Marginal Shares (β_i) (2)	Income elasticity (η_i) (3)	Marshallian price elasticity (uncompensated) (η^*_{ij}) (4)	Slutsky price elasticity (compensated (η_{ij}) (5)
Food	0.078	0.186	-0.146	-0.068
Beverages and tobacco	0.009	0.357	-0.149	-0.140
Housing	0.108	0.684	-0.349	-0.241
Fuel and lighting	0.015	0.415	-0.176	-0.161
Personal effects	0.007	0.366	-0.150	-0.143
Health	0.026	1.291	-0.522	-0.496
Transportation	0.094	1.388	-0.590	-0.496
Communication	0.014	0.820	-0.333	-0.319
Education	0.037	1.277	-0.522	-0.485
Recreation	0.022	2.032	-0.807	-0.785
Clothing	0.013	0.494	-0.205	-0.192
Durables	0.220	5.098	-1.790	-1.570
Other	0.357	2.839	-1.078	-0.721

A New Method to Estimate σ_h

The following is the quote from Frisch (1959):

We may, perhaps, assume that in most cases the [Frisch parameter] has values of the order of magnitude given below.

$\sigma_h = -10$ *for the extremely poor and apathetic part of the population.*

$\sigma_h = -4$ *for the slightly better off but still poor part of the population with a fairly pronounced desire to become better off.*

$\sigma_h = -2$ *for the middle-income bracket, "the median part" of the population.*

$\sigma_h = -0.7$ *for the better off part of the population.*

$\sigma_h = -0.1$ *for the rich part of the population with ambitions towards "conspicuous consumption".*

The Quadratic Expenditure System (QES)

- The QES (Pollak and Wales, 1978) is based on the following indirect utility function.

$$U_I(M, \mathrm{p}_1, \mathrm{p}_2, \ldots, \mathrm{p}_\mathrm{n}) = -\frac{\prod_{k=1}^{n} {p_k}^{\beta_k}}{(M-\sum_{k=1}^{n} p_k\gamma_k)} + \mu \frac{\prod_{k=1}^{n} {p_k}^{\beta_k}}{\prod_{k=1}^{n} {p_k}^{c_k}}$$

where

$$\sum_{i=1}^{n} \beta_i = 1 \qquad \text{and} \qquad \sum_{i=1}^{n} c_i = 1.$$

- Roy's (1942) theorem:

$$q_i = -\frac{\partial U_I/\partial p_i}{\partial U_I/\partial M}$$

The Quadratic Expenditure System (QES)

- QES given in the form

$$p_i q_i = p_i \gamma_i + \beta_i \left(M - \sum_{k=1}^{n} p_k \gamma_k \right) + \mu (c_i - \beta_i) \prod_{k=1}^{n} p_k^{-c_k} \left(M - \sum_{k=1}^{n} p_k \gamma_k \right)^2,$$

can be written as

$$p_i q_i = A_{1i} + A_{2i} M + A_{3i} M^2,$$

where

$$\mathrm{A_{3i}} = \mu (c_i - \beta_i) \left(\prod_{k=1}^{n} p_k^{-c_k} \right)$$

$$A_{1i} = p_i \gamma_i - \beta_i \left(\sum_{j=1}^{n} p_j \gamma_j \right) + \mu (c_i - \beta_i) \left(\prod_{k=1}^{n} p_k^{-c_k} \right) \left(\sum_{j=1}^{n} p_j \gamma_j \right)^2 = p_i \gamma_i - \beta_i \left(\sum_{j=1}^{n} p_j \gamma_j \right) + A_{3i} \left(\sum_{j=1}^{n} p_j \gamma_j \right)^2 \quad (4)$$

$$A_{2i} = \beta_i - 2\mu (c_i - \beta_i) \left(\prod_{k=1}^{n} p_k^{-c_k} \right) \left(\sum_{j=1}^{n} p_j \gamma_j \right) = \beta_i - 2 A_{3i} \left(\sum_{j=1}^{n} p_j \gamma_j \right). \quad (5)$$

The Quadratic Expenditure System (QES)

- Income flexibility

$$(M - \sum_{k=1}^{n} p_k \gamma_k) = -2M\phi \text{, and } \sum_{k=1}^{n} s_k = 1 + 2\phi$$

$$A_{1i} = p_i\gamma_i - \beta_i\left(\sum_{j=1}^{n} p_j\gamma_j\right) + \mu(c_i - \beta_i)\left(\prod_{k=1}^{n} p_k^{-c_k}\right)\left(\sum_{j=1}^{n} p_j\gamma_j\right)^2 = p_i\gamma_i - \beta_i\left(\sum_{j=1}^{n} p_j\gamma_j\right) + A_{3i}\left(\sum_{j=1}^{n} p_j\gamma_j\right)^2 \quad (4)$$

$$A_{2i} = \beta_i - 2\mu(c_i - \beta_i)\left(\prod_{k=1}^{n} p_k^{-c_k}\right)\left(\sum_{j=1}^{n} p_j\gamma_j\right) = \beta_i - 2A_{3i}\left(\sum_{j=1}^{n} p_j\gamma_j\right). \quad (5)$$

From (5), we have

$$\beta_i = A_{2i} + 2A_{3i}M(1 + 2\phi)$$

and from (4) we have

$$s_i = p_i\gamma_i / M = \frac{1}{M} A_{1i} + \beta_i(1 + 2\phi) - A_{3i}M(1 + 2\phi)^2$$

The Quadratic Expenditure System (QES)

$$p_i q_i = p_i \gamma_i + \beta_i \left(M - \sum_{k=1}^{n} p_k \gamma_k \right) + \mu (c_i - \beta_i) \prod_{k=1}^{n} p_k^{-c_k} \left(M - \sum_{k=1}^{n} p_k \gamma_k \right)^2 ,$$

Income flexibility

$(M - \sum_{k=1}^{n} p_k \gamma_k) = -2M\phi$, and $\sum_{k=1}^{n} s_k = 1 + 2\phi$

$$w_i = s_i + B1_i\, \phi + B2_i\, (\phi)^2, \qquad i=1,2,\ldots,n. \qquad (6)$$

where $B1_i = -2(A_{2i} + 2A_{3i}M)$ and $B2_i = -4A_{3i}M$.

The Quadratic Expenditure System (QES)

Income and price elasticities under QES:

$$\eta_i = \frac{\beta_i}{w_i} + \frac{2}{w_i} A_{3i} \left(M - \sum_{j=1}^{n} p_j \gamma_j \right) = \frac{1}{w_i} (\beta_i - 4\phi M A_{3i});$$

$$\eta_{ij}^{*} = \delta_{ij} \left(\frac{s_i}{w_i} - 1 \right) - \frac{\beta_i}{w_i} s_j + \frac{4\phi M A_{3i}}{w_i} (s_j - c_j \phi);$$

$$\eta_{ij} = \delta_{ij} \left(\frac{s_i}{w_i} - 1 \right) - \frac{\beta_i}{w_i} s_j + \frac{4\phi M A_{3i}}{w_i} (s_j - c_j \phi) - \frac{w_j}{w_i} (\beta_i - 4\phi M A_{3i}).$$

where $s_i = p_i \gamma_i / M$ and $A_{3i} = \mu(c_i - \beta_i) \left(\prod_{k=1}^{n} p_k^{-c_k} \right).$

The Quadratic Expenditure System (QES)

Step 1: Estimate QES in the form $p_i q_i = A_{1i} + A_{2i}M + A_{3i}M^2$, and obtain estimates for A_{1i}, A_{2i} and A_{3i} for i=1,2, …, n.

Step 2: Using the estimated values of A_{1i}, A_{2i} and A_{3i} for i=1,2, …, n, estimate Equation below to obtain estimate for ϕ.

$$w_i = s_i + B1_i \phi + B2_i (\phi)^2$$

Step 3: Use the estimates obtained for A_{1i}, A_{2i} and A_{3i} for i=1,2, …, n. in Step 1 and ϕ in Step 2 in Equation below to calculate estimates for β_i, i=1,2, .,, .n.

$$\beta_i = A_{2i} + 2A_{3i}M(1+2\phi)$$

The Quadratic Expenditure System (QES)

Step 4: Use the estimates obtain for A_{1i}, A_{2i} and A_{3i} for i=1,2, …, n. is Step 1 and ϕ in Step 2 and for β_i, i=1,2, …,n, in Step 3, to calculate s_i for i=1,2, …,n, from Equation below.

$$s_i = p_i \gamma_i / M = \frac{1}{M} A_{1i} + \beta_i (1 + 2\phi) - A_{3i} M (1 + 2\phi)^2$$

Step 5: Use the estimates obtain for A_{1i}, A_{2i} and A_{3i} for i=1,2, …, n. is Step 1 and ϕ in Step 2 and for β_i, i=1,2, …,n. in Step3, and calculated value of s_i, i=1,2, …, n, in Step 4 to calculate the income and price elasticities.

The Quadratic Expenditure System (QES)

Step 5 (cont): To calculate the price elasticities, we need values of C_i, i=1,2,…,n, subject to the restriction $\sum_{i=1}^{n} C_i$=1. Without prices, these parameters are not identified. We generate 100 sets of random numbers subject to this restriction and calculate 100 sets of price elasticities and calculate the average of the 100 sets of price elasticities.

The Quadratic Expenditure System (QES)

Table 3: Estimates of income and price elasticities based on QES Estimation Results, Sri Lanka, HES (2019) data

Commodity	Budget share	Income elasticity	Marshallian uncompensated	Slutsky compensated
Food	0.443	0.300	-0.254	-0.387
Beverages & Tobacco	0.034	0.420	-0.376	-0.390
Housing	0.149	0.968	-0.829	-0.974
Fuel & Lighting	0.040	0.547	-0.467	-0.489
Personal effects	0.019	0.485	-0.428	-0.438
Health	0.024	1.882	-1.623	-1.669
Transportation	0.064	1.681	-1.438	-1.545
Communication	0.017	1.146	-0.997	-1.017
Education	0.031	1.585	-1.380	-1.430
Recreation	0.011	2.485	-2.041	-2.067
Clothing	0.029	0.706	-0.618	-0.639
Durables	0.036	4.570	-3.678	-3.843
Other	0.104	2.362	-1.874	-2.119

Estimate of ϕ = -0.42 (0.06); and σ = -2.60 (0.06).